Lecture 15 (5. May. 2023)

Today we will review some of the most relevant aspects of the theory of modular forms covered in this course.

(1) The modular group and its action on $\mathbb{H}$

$GL_2(\mathbb{C})$ acts on $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

and this restricts to an action of $GL_2^+(\mathbb{R})$ on $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$

(the $+$ means $\det(\ ) > 0$)

The action of $GL_2^+(\mathbb{R})$ on $\mathbb{H}$ is transitive. Indeed: if $x+iy \in \mathbb{H}$ then

$$x+iy = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix} \cdot i$$

so any point in $\mathbb{H}$ is equivalent under $GL_2^+(\mathbb{R})$ to i (hence $GL_2^+(\mathbb{R}) \backslash \mathbb{H}$ is just one point)

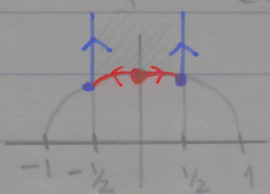
The modular group is $\Gamma = SL_2(\mathbb{Z}) \subseteq GL_2^+(\mathbb{R})$ (discrete subgroup)

Thm 4.(d): $\Gamma = \left\langle T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$

Thm 4.(a)-(b): $\Gamma \backslash \mathbb{H}$ can be identified with $F/\sim$ where $F := \{z \in \mathbb{H} : -1/2 \leq \operatorname{Re}(z) \leq 1/2,\ |z| \geq 1\}$ and for $z \in F$

$z \sim z+1$ if $\operatorname{Re}(z) = -1/2$, $z \sim -\frac{1}{z} = -\bar{z}$ if $|z| = 1$

I.e.



Note that F in unbounded, but it has finite hyperbolic volume: $d\mu = \frac{dx\,dy}{y^2}$ is the hyperbolic measure

It is invariant under the action of $GL_2^+(\mathbb{R})$, i.e.

$$\mu(A) := \int_A 1 \frac{dx\,dy}{y^2} = \mu(g\cdot A) \quad \forall g \in GL_2^+(\mathbb{R})$$
$\forall A \subseteq \mathbb{H}$ Borel subset

We have
$$\mu(F) = \int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{\infty} 1 \frac{dy\,dx}{y^2} = \int_{-1/2}^{1/2} \left(-\frac{1}{y}\right)\Big|_{y=\sqrt{1-x^2}}^{y=\infty} dx$$
$$= \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = 2\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$$
$$= 2\int_0^{\pi/6} \frac{\cos(\theta)\,d\theta}{\sqrt{1-\sin(\theta)^2}} = \frac{\pi}{3}$$

Hence $\mu(\Gamma\backslash\mathbb{H}) = \pi/3$

Rmk: $\Gamma\backslash\mathbb{H}$ is also a Riemann surface (connected one dim. complex manifold), non compact.

Compactification of $\Gamma\backslash\mathbb{H}$

Put $\overline{\mathbb{H}} := \mathbb{H} \cup (\mathbb{Q} \cup \{\infty\})$

Γ also acts on $\mathbb{Q}\cup\{\infty\}$ and the action is transitive:

If $\frac{p}{q} \in \mathbb{Q}$, $p,q\in\mathbb{Z}$ with $\gcd(p,q)=1$ then by Bézout's Lemma $\exists\, b,d \in \mathbb{Z}$ such that $pd-qb=1$ hence $\begin{pmatrix} p & b \\ q & d\end{pmatrix}\cdot\infty = \frac{p}{q}$.

Hence $\Gamma\backslash\overline{\mathbb{H}} = (\Gamma\backslash\mathbb{H}) \cup \underbrace{\{\infty\}}_{\Gamma\backslash(\mathbb{Q}\cup\{\infty\})}$. This is a compactification of $\Gamma\backslash\mathbb{H}$.

2 Modular forms for $SL_2(\mathbb{Z})$

Defn: For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$, $k \in \mathbb{Z}$, $f : \mathbb{H} \to \mathbb{C}$ we define

$$(f|_k \gamma)(z) = \det(\gamma)^{k/2} (cz+d)^{-k} f(\gamma \cdot z)$$

This defines a right action of $GL_2^+(\mathbb{R})$ on functions $f : \mathbb{H} \to \mathbb{C}$.

A mod. form of weight k for $SL_2(\mathbb{Z})$ is a function $f : \mathbb{H} \to \mathbb{C}$ such that
i) f is holom in $\mathbb{H}$
ii) $f|_k \gamma = f \quad \forall \gamma \in SL_2(\mathbb{Z})$
iii) f is holomorphic at ∞, i.e. $f(z) = \sum_{n=0}^{\infty} a_n q^n$, $q := e^{2\pi i z}$

Sometimes we write $a_n = a_n(f)$ = the nth Fou. coeff of f.

$M_k := \{$ mod. forms weight k for $SL_2(\mathbb{Z})\}$

$S_k := \{ f \in M_k : a_0(f) = 0 \}$ subspace of cusp forms

Examples: i) Constant functions are mod. forms of weight zero

In fact $M_0 = \mathbb{C}$ ($f \in M_0 \Rightarrow f$ induces a holomorphic function on $\overline{\Gamma \backslash \mathbb{H}}$, but on a compact Riemann surface the only functions that are holom. everywhere are constants)

ii) If k is odd then $M_k = \{0\}$

iii) Eisenstein series: If $k \geq 4$ is even integer then

$$G_k(z) = \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} \frac{1}{(mz+n)^k} \quad \text{is in } M_k \setminus S_k$$

Prop 2.2: $G_k(z) = 2\zeta(k) + 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$

$E_k(z) := \frac{1}{2\zeta} G_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$ is the normalized Eis. series of wt. k

iv) The discriminant mod. form : $\Delta := \dfrac{E_4^3 - E_6^2}{1728} \in S_{12}$

3 Structure of M_k and S_k

M_k is a $\mathbb{C}$-vector space of fin. dimension and S_k is a $\mathbb{C}$-vector subspace

Thm 3.5 Let $k \in 2\mathbb{Z}$

(i) $M_0 = \mathbb{C}$

(ii) $M_k = \{0\}$ if $k < 0$ or $k = 2$

(iii) $M_k = \mathbb{C} E_k$ if $k = 4, 6, 8, 10, 14$

(iv) $S_k = \{0\}$ if $k < 12$ or $k = 14$

$S_{12} = \mathbb{C}\Delta$

$S_k = \Delta \cdot M_{k-12}$ if $k \geq 16$

(v) $M_k = \mathbb{C}E_k \oplus S_k$ if $k \geq 4$

Cor Let $k \geq 0$ even. Then $\dim_{\mathbb{C}} M_k = \begin{cases} \lfloor k/12 \rfloor + 1 & \text{if } k \not\equiv 2 \pmod{12} \\ \lfloor k/12 \rfloor & \text{if } k \equiv 2 \pmod{12} \end{cases}$

Also $S_k = \{0\}$ if $k < 12$ and for $k \geq 12$, $\dim_{\mathbb{C}} S_k = \dim_{\mathbb{C}} M_k - 1$.

We also have

Prop 3.6 : $M_* := \bigoplus_{k \in \mathbb{Z}} M_k = \mathbb{C}[E_4, E_6] \simeq \mathbb{C}[X, Y]$ as $\mathbb{C}$-algebras

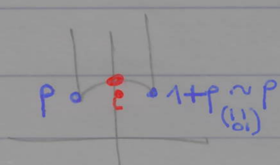
(because E_4, E_6 are alg. indep)

Thm 3.5 is a consequence of the "valence formula" (Lecture 6)

$$\sum_{p \in \Gamma \backslash \mathbb{H}} \frac{1}{e_p} \cdot \mathrm{ord}_p(f) + \mathrm{ord}_\infty(f) = \frac{k}{12}$$

for any (meromorphic) modular form $f \neq 0$ of weigh k

where $e_p = \begin{cases} 3 & \text{if } p \underset{\Gamma}{\sim} e^{2\pi i/3} = \rho \\ 2 & \text{if } p \underset{\Gamma}{\sim} i \\ 1 & \text{otherwise} \end{cases}$



Ex : One can use the valence formula to show that $\Delta(z) \neq 0 \ \forall z \in \mathbb{H}$. Indeed, we know

$$\Delta(z) = \frac{E_4^3(z) - E_6^2(z)}{1728} = q - 24q^2 + 252q^3 + \dots = \sum_{n=1}^{\infty} \tau(n) q^n$$

hence $\mathrm{ord}_\infty \Delta = 1$, and since $k = 12$ we must have $\mathrm{ord}_p(\Delta) = 0 \ \forall p \in \Gamma \backslash \mathbb{H}$.

The fact that Δ has no zeroes in $\mathbb{H}$ is also a consequence of $\Delta(z) = q \cdot \prod_{n=1}^{\infty} (1-q^n)^{24}$ (ES 3, Exercise 2)

Since $q = e^{2\pi i z} \neq 0$ and $|q| = e^{-2\pi \mathrm{Im}(z)} < 1$.

4 Modular L-functions

Inspired on $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ and $L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ where χ is a Dirichlet character (i.e. $\chi: \mathbb{Z}^+ \to \mathbb{C}$ induced by group character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$ with $\chi(n) = 0$ if $\gcd(n,N) > 1$) one defines: given $f \in M_k$, $f(z) = \sum_{n=1}^{\infty} a_n q^n$ its L-function is $L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$

For this to converge for some value of $s \in \mathbb{C}$ we need bounds for a_n (in terms of n). From Lecture 8:

i) if $f = E_k$ then $\exists A, B > 0$ s.t. $A n^{k-1} \leq |a_n| \leq B n^{k-1}$

ii) if $f \in S_k$ then $a_n = O(n^{k/2})$, meaning that $\exists C > 0$ such that $|a_n| \leq C n^{k/2} \ \forall n \geq 1$.

In the first case $L(f,s)$ conv. abs. and unif. in $\mathrm{Re}(s) \geq k + \varepsilon$ for all $\varepsilon > 0$, and in the second case in $\mathrm{Re}(s) \geq \frac{k}{2} + 1 + \varepsilon$ for all $\varepsilon > 0$.

Thm 4.2 $L(f,s)$ has mero. cont. to all of $s \in \mathbb{C}$ satisfying

$$\Lambda(f,s) := (2\pi)^{-s} \Gamma(s) L(f,s) = (-1)^{k/2} \Lambda(f, k-s)$$

Moreover, $\Lambda(f,s) + \frac{a_0}{s} + \frac{(-1)^{k/2} a_0}{k-s}$ is entire.

There is also a "converse theorem" of Hecke (Theorem 4.3.8 in Miyake's book "Modular Forms"): If $(a_n)_{n\geq 0}$ is a seq. in $\mathbb{C}$ with $a_n = O(n^c)$ for some $c>0$ and the function

$$\Lambda(s) = (2\pi)^{-s}\,\Gamma(s)\sum_{n=1}^{\infty}\frac{a_n}{n^s}, \quad \text{for } s\in\mathbb{C} \text{ with } \mathrm{Re}(s)>c+1,$$

has analytic cont. to $\mathbb{C}$ satisfying $\Lambda(k-s) = (-1)^{k/2}\Lambda(s)$, bounded on vertical strips as $\mathrm{Im}(s)\to\pm\infty$ and $\Lambda(s) + \frac{a_0}{s} + \frac{(-1)^{k/2}a_0}{k-s}$ is entire, then

$$f(z) := \sum_{n=0}^{\infty} a_n q^n \text{ is in } M_k.$$

Example (ES 4, Exercise 2): If $f = E_k = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$

then $L(f,s) = \left(\frac{-2k}{B_k}\right)\cdot\sum_{n=1}^{\infty}\frac{\sigma_{k-1}(n)}{n^s} = \left(\frac{-2k}{B_k}\right)\zeta(s)\zeta(s-k+1)$ //

Comment: Many properties refering to generalitations of modular forms (automorphic forms / representations) are stated via L-functions.

5 Petersson inner product

Recall $d\mu = \frac{dx\,dy}{y^2}$ the hyp. measure in $\mathbb{H}$ inv. under $GL_2^+(\mathbb{R})$

Given $f, g \in S_k$ one defines $\langle f, g\rangle = \int_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)}\,y^k\,d\mu(z)$

Rmk: $\phi(z) := f(z)\overline{g(z)}\,y^k$ is Γ-inv (i.e $\phi(\gamma z) = \phi(z)\ \forall \gamma\in\Gamma$) hence it is meaningful to integrate ϕ over $\Gamma\backslash\mathbb{H}$. Also:

$$\int_{\Gamma\backslash\mathbb{H}}\phi(z)\,d\mu(z) = \int_{F}\phi(x+iy)\,\frac{dx\,dy}{y^2}$$

If $f = E_k$, $g\in S_k$ then $\langle E_k, g\rangle$ also makes sense because the integral converges, but a computation gives $\langle E_k, g\rangle = 0$

6 Hecke operators

For $n \geq 1$ integer: $M(n) = \{\gamma \in M_{2\times 2}(\mathbb{Z}) : \det(\gamma) = n\}$

$$T_n : M_k \longrightarrow M_k$$
$$f \longmapsto \sum_{\gamma \in \Gamma \backslash M(n)} f|_k \gamma$$

Using that $R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = n, \ b \in \{0, 1, \ldots, d-1\} \right\}$ is a set of rep's for $\Gamma \backslash \Gamma(n)$ we get:

$$T_n(f) = \frac{1}{n} \sum_{ad=n} a^{k-1} \sum_{b \bmod d} f\left(\frac{az+b}{d}\right)$$

This can be used to prove $T_n(E_k) = \sigma_{k-1}(n) \cdot E_k \ \forall n \geq 1$

Properties of Hecke operators:

i) $T_n(S_k) \subseteq S_k$

ii) If $f = \sum_{n=0}^{\infty} c_n q^n$, $T_m(f) = \sum_{n=0}^{\infty} b_n q^n$ then

$$b_n = \begin{cases} c_0 \cdot \sigma_{k-1}(m) & \text{if } n = 0, \\ \sum_{d/(n,m)} d^{k-1} c_{\frac{nm}{d^2}} & \text{if } n \geq 1 \end{cases}$$

iii) $T_n T_m = \sum_{d/(n,m)} d^{k-1} T_{\frac{nm}{d^2}}$

iv) $\forall n \geq 1$, T_n is self-adjoint on S_k, i.e.

$$\langle T_n f, g \rangle = \langle f, T_n g \rangle \quad \forall f, g \in S_k$$

v) $\exists$ basis of S_k formed by normalized (1-st coeff $= 1$) simultaneous eigenforms for all the T_n's. If f is such a form then $f = \sum_{n=1}^{\infty} a_n q^n$ with: $T_n(f) = a_n f$

$$a_n \cdot a_m = \sum_{d/(n,m)} d^{k-1} a_{\frac{nm}{d^2}}$$

This implies: $a_n a_m = a_{nm}$ if $\gcd(n,m) = 1$

$$a_{p^r} a_p = a_{p^{r+1}} + p^{k-1} a_{p^{r-1}} \quad \forall \text{ prime } p \ \forall r \geq 1$$